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The relationship between the dispersionless equation and the localized induction hierarchy through the Pohlmeyer–Lund–Regge equation

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Abstract. Fukumoto and Miyajima (Fukumoto Y and Miyajima M 1996 *J. Phys. A: Math. Gen.* **29** 8025) have discussed that an evolution equation obtained by summing up an infinite number of equations of the localized induction hierarchy (LIH) is equivalent to the Pohlmeyer–Lund–Regge equation (PLRE). On the other hand, the dispersionless equation (DLE) is also shown to be equivalent to the PLRE, so we investigate a link between the equation derived from the LIH and the DLE through the PLRE. We find that their result is not correct. However, we show that there is a special case where a solution of the LIH satisfies the PLRE.

1. Introduction

Connections among integrable equations are an interesting subject. Fukumoto and Miyajima [1] have studied the relationship between the sum of an infinite number of equations of the localized induction hierarchy (LIH) and the Pohlmeyer–Lund–Regge equation (PLRE) [2, 3] and have concluded that the LIH is equivalent to the PLRE. On the other hand, Imai *et al* [4] have shown that the dispersionless equation (DLE) [5] is equivalent to the PLRE in a simple way. Furthermore, they have proposed a system with two hierarchies, one of which is the LIH and the other one includes the DLE [6]. We note that the PLRE has two different representations, that is, (8) and (9) with $C = 0$ and (18) and that they are proved to be equivalent to each other [7].

In this paper we demonstrate that the result of Fukumoto and Miyajima is not correct. However, we show that there exists a solution of the LIH satisfying the PLRE by using one soliton solution of the system with two hierarchies.

In the next section we review the paper of Fukumoto and Miyajima and demonstrate that their condition of equivalence between the LIH and the PLRE is not satisfied except for a very special case. A simple discussion on the connection between the DLE and the PLRE is given in section 3. We discuss the connection between the LIH and the DLE through the soliton solution of the system with two hierarchies in section 4. The last section is devoted to the conclusion.

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2. Localized induction hierarchy and PLR equation

The localized induction equation is given by

$$\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} + \alpha(\boldsymbol{X}_{sss} + \tfrac{3}{2}\boldsymbol{X}_{ss} \times (\boldsymbol{X}_s \times \boldsymbol{X}_{ss})) \tag{1}$$

where $\boldsymbol{X}$ means a position vector on the thin vortex filament and s denotes the arclength along the filament. Here, the first term of the RHS represents the swirl flow and the second one the axial flow [8]. Fukumoto and Miyajima considered the following LIH:

$$\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \varepsilon \boldsymbol{V}^{(2)} + \varepsilon^2 \boldsymbol{V}^{(3)} + \cdots = \sum_{n=1}^{\infty} \varepsilon^{n-1} \boldsymbol{V}^{(n)} \tag{2}$$

where $\boldsymbol{V}^{(n)}$s are given by [9]

$$\begin{aligned} \boldsymbol{V}^{(1)} &= \boldsymbol{X}_s \times \boldsymbol{X}_{ss} \\ \boldsymbol{V}^{(2)} &= -\boldsymbol{X}_s \times \boldsymbol{V}_s^{(1)} + \mathcal{T}^{(2)}\boldsymbol{X}_s \\ &\cdots \\ \boldsymbol{V}^{(n)} &= -\boldsymbol{X}_s \times \boldsymbol{V}_s^{(n-1)} + \mathcal{T}^{(n)}\boldsymbol{X}_s \\ &\cdots . \end{aligned} \tag{3}$$

Then summing up an infinite number of equations of the hierarchy, we obtain

$$\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} - \varepsilon \boldsymbol{X}_s \times \boldsymbol{X}_{ts} + \mathcal{T}\boldsymbol{X}_s. \tag{4}$$

Taking the inner product with $\boldsymbol{X}_s$ and the derivative of (4) with respect to s, we obtain

$$\mathcal{T} = \frac{\varepsilon}{2}\boldsymbol{X}_t \cdot \boldsymbol{X}_t + C(t) \tag{5}$$

where $C(t)$ is an arbitrary function of t obtained by integration with respect to s. Taking the outer product of (4) with $\boldsymbol{X}_s$, we have

$$\boldsymbol{X}_s \times \boldsymbol{X}_t = -\boldsymbol{X}_{ss} + \varepsilon \boldsymbol{X}_{st}. \tag{6}$$

By changing variables

$$\zeta = s \qquad \eta = \frac{2t}{\varepsilon} + s \tag{7}$$

equation (6) yields

$$\boldsymbol{X}_{\zeta\zeta} - \boldsymbol{X}_{\eta\eta} = -\frac{2}{\varepsilon}\boldsymbol{X}_\zeta \times \boldsymbol{X}_\eta. \tag{8}$$

From $\boldsymbol{X}_s^2 = 1$, we obtain

$$\boldsymbol{X}_\zeta^2 + \boldsymbol{X}_\eta^2 = 1 - \varepsilon C(t) \qquad \boldsymbol{X}_\zeta \cdot \boldsymbol{X}_\eta = \frac{\varepsilon}{2}C(t). \tag{9}$$

Fukumoto and Miyajima have concluded that, if $C(t) = 0$, (8) and (9) are just the PLRE [3].

At first glance, $C(t)$ seems to be completely arbitrary, since it is an integration constant. However, we find that, by taking the inner product of (4) with $\boldsymbol{X}_s$ by referring to (5), C is determined as

$$\begin{aligned} C &= \boldsymbol{X}_t \cdot \boldsymbol{X}_s - \frac{\varepsilon}{2}\boldsymbol{X}_t \cdot \boldsymbol{X}_t \\ &= \boldsymbol{X}_t \cdot \boldsymbol{X}_\eta. \end{aligned} \tag{10}$$

Of course, $C_s = 0$. Since C depends on $\boldsymbol{X}_s$ and $\boldsymbol{X}_t$, the condition $C = 0$ is not satisfied automatically. Therefore we say that the LIH is not equivalent to the PLRE in general.

In section 4 we show, again, that the LIH is not equivalent to the PLRE by using the fact of the equivalence between the PLRE and the DLE. However, this does not means that all of the solutions of the LIH do not satisfy the PLRE. Therefore we show the case where one soliton solution of the LIH satisfies the PLRE.

3. Dispersionless equation and PLR equation

We introduce the following system:

$$\frac{\partial \boldsymbol{R}}{\partial t} = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & \mu_1 \\ \mu_2 & -\mu_1 & 0 \end{pmatrix} \boldsymbol{R} \tag{11}$$

where $\boldsymbol{R} = (R_1, R_2, R_3)^{\mathrm{t}}$. Assuming

$$\mu_{1x} = 2R_2 \qquad \mu_{2x} = -2R_1 \qquad \mu_{3x} = 0 \tag{12}$$

and taking

$$R_1 = X_{1s} \qquad R_2 = X_{2s} \qquad R_3 = X_{3s} \tag{13}$$

we can obtain the DLE

$$X_{1st} - 2X_{3s}X_1 = 0 \qquad X_{2st} - 2X_{3s}X_2 = 0 \qquad X_{3st} + 2X_{1s}X_1 + 2X_{2s}X_2 = 0 \tag{14}$$

which is integrable and solved by the inverse scheme [5].

With the rotational angles γ, ϕ, ν of O(3) such as

$$\boldsymbol{R} = \mathrm{e}^{-\gamma J_3}\mathrm{e}^{-\phi J_2}\mathrm{e}^{-\nu J_3}\boldsymbol{R}_0 \tag{15}$$

with $\boldsymbol{R}_0 = (0, 0, 1)^{\mathrm{t}}$, we have

$$\phi_{st} - \gamma_s\gamma_t \tan\phi = 2\sin\phi \qquad \gamma_{st} + \phi_t\gamma_s \cot\phi + \frac{\gamma_t\phi_s}{\sin\phi\cos\phi} = 0 \tag{16}$$

by eliminating ν. Changing variables from (ϕ, γ) to (ϕ, θ) as

$$\gamma_t = \frac{\theta_t \cos\phi}{2\cos^2\frac{\phi}{2}} \qquad \gamma_s = \frac{\theta_s}{2\cos^2\frac{\phi}{2}} \tag{17}$$

we obtain the PLRE

$$\phi_{st} - \frac{\theta_t\theta_s \tan\frac{\phi}{2}}{2\cos^2\frac{\phi}{2}} = 2\sin\phi \qquad \theta_{st} + \frac{\theta_s\phi_t + \theta_t\phi_s}{\sin\phi} = 0. \tag{18}$$

So, the DLE is equivalent to another form of the PLRE (18) [2, 3], which is equivalent to (8) and (9) with $C = 0$ [7].

4. Localized induction hierarchy and dispersionless equation

In order to discuss the relationship between the LIH and the DLE, we have proposed the following inverse scheme [6]:

$$V_s = UV \qquad V_t = WV \tag{19}$$

where

$$U = \lambda R = \sum_a \lambda R_a T^a \qquad W = \sum_{n=-\infty}^{\infty} \lambda^n W_n = \sum_n \sum_a \lambda^n W_{an} T^a. \tag{20}$$

Here, T^a is a generator of the Lie algebra and R and W_n are vectors. This system has a conserved quantity

$$\mathrm{Tr}\, R^2. \tag{21}$$

The compatibility condition of (19) is given by

$$U_t - W_s + [U, W] = 0. \tag{22}$$

Equation (22) gives the following series of equations ordered by the powers of λ:

$$
\begin{aligned}
&\cdots \\
&-W_{3s} + [R, W_2] = 0 \\
&-W_{2s} + [R, W_1] = 0 \\
&R_t - W_{1s} + [R, W_0] = 0 \\
&-W_{0s} + [R, W_{-1}] = 0 \\
&-W_{-1s} + [R, W_{-2}] = 0 \\
&\cdots .
\end{aligned}
\tag{23}
$$

The equation of motion for R_t consists of two hierarchies. One is the upper hierarchy determined by W_1, from which the LIH is derived if we assume $su(2)$ as the Lie algebra, and another the lower hierarchy obtained by W_0 where the DLE is included. For details see [6]. We obtain the equation of motion

$$
\begin{aligned}
R_t - A_1 R_s + \frac{A_2}{2}[R, R_s]_s + A_3 \left(R_{ss} + \frac{3}{8}[R_s, [R, R_s]] \right)_s \cdots \\
+[R, B_0] + \left[R, \int^s [R, B_{-1}]\, \mathrm{d}s' \right] \\
+\left[R, \int^s \left[R, \int^{s'} [R, B_{-2}]\, \mathrm{d}s'' \right] \mathrm{d}s' \right] + \cdots = 0
\end{aligned}
\tag{24}
$$

where A_i is a constant and B_i is a constant matrix. With the vector notation $\boldsymbol{X}_s = \boldsymbol{R}$, (24) reduces to

$$
\begin{aligned}
\boldsymbol{X}_{st} - A_1 \boldsymbol{X}_{ss} + A_2 \boldsymbol{X}_s \times \boldsymbol{X}_{sss} + A_3[\boldsymbol{X}_{sss} + \tfrac{3}{2} \boldsymbol{X}_{ss} \times (\boldsymbol{X}_s \times \boldsymbol{X}_{ss})]_s \\
+ \cdots + 2\boldsymbol{X}_s \times \boldsymbol{B}_0 + 4\boldsymbol{X}_s \times (\boldsymbol{X} \times \boldsymbol{B}_{-1}) \\
-8\boldsymbol{X}_s \times \left(\int^s \boldsymbol{X}_{s'} \times (\boldsymbol{X} \times \boldsymbol{B}_{-2})\, \mathrm{d}s' \right) + \cdots = 0.
\end{aligned}
\tag{25}
$$

Here, $\boldsymbol{B}_i$ denotes a constant vector. The upper part of the hierarchies, that is, the LIH, represents that the term with the coefficient A_1 denotes the slipping motion of the thin vortex filament with the speed A_1, the next term the swirl flow and the term with A_3 the axial flow. The lower part of the hierarchies shows that the term with $\boldsymbol{B}_0$ represents the velocity of the vortex filament in the rotating fluid with the angular velocity $\boldsymbol{B}_0$ and the next one, that is, the DLE, represents a correction if the angular velocity has a dependence on the position in such a way as $\boldsymbol{B}_0 + \boldsymbol{X} \times \boldsymbol{B}_{-1}$.

We can see that, since the LIH belongs to members of the upper part of the hierarchies and the DLE belongs to those of the lower one, they are members within the hierarchies so the LIH and the DLE are independent from each other in the sense that they are derived from involutive conserved densities. This fact means that the LIH is not equivalent to the PLRE because of equivalence between the DLE and the PLRE.

With the inverse method, we can obtain soliton solutions. Details will be published in a separate paper [10]. One soliton solution has the following form:

$$
\boldsymbol{X}_s(s, t) = \boldsymbol{X}_s(\lambda X_0 s - \omega t) \tag{26}
$$

where λ is a spectrum parameter of the inverse method and X_0 is determined by the asymptotic behaviour of $\boldsymbol{X}$. ω is given by

$$
\omega = (-\lambda A_1 + 2A_2\lambda^2 - 4A_3\lambda^3 + \cdots)X_0 + B_0 - \frac{B_{-1}}{\lambda} + \cdots . \tag{27}
$$

The LIH of (2) is given if we take

$$A_1 = 0 \qquad A_2 = 1 \qquad A_3 = -\varepsilon, \ldots \tag{28}$$

and $\boldsymbol{B}_i = 0$, $(i = 0, -1, -2, \ldots)$. Summing up the A_is in ω, we obtain

$$\omega_{\mathrm{LIH}} = \frac{2\lambda^2}{1 - 2\varepsilon\lambda} \tag{29}$$

which means that the soliton solution for the LIH is given by

$$\boldsymbol{X}_s(s, t) = \boldsymbol{X}_s(\lambda X_0 s - \omega_{\mathrm{LIH}} t). \tag{30}$$

If (29) is equal to $-B_{-1}/\lambda$, then the solution of LIH is equal to that of the DLE:

$$\boldsymbol{X}_s(s, t) = \boldsymbol{X}_s\left(\lambda X_0 s + \frac{B_{-1}}{\lambda} t\right) \tag{31}$$

so that (31) is equivalent to one soliton solution of PLRE.

5. Conclusion

We have considered a relationship of integrable equations among the LIH, PLRE and DLE. The equivalence between the PLRE and DLE is proved in a very simple way with the angular variables in the rotational space. We have found that the integration constant C is not arbitrary, but it depends on $\boldsymbol{X}_s$ and $\boldsymbol{X}_t$, so the LIH has been shown not to be equivalent to the PLRE. Furthermore, by using the system with two hierarchies, nonequivalence between the LIH and the PLRE has been made clear by the help of the DLE. However, we have found a solution of the LIH satisfying the PLRE.

References

[1] Fukumoto Y and Miyajima M 1996 *J. Phys. A: Math. Gen.* **29** 8025
[2] Pohlmeyer K 1976 *Commun. Math. Phys.* **46** 207
[3] Lund F and Regge T 1976 *Phys. Rev.* D **14** 1524
[4] Imai K, Konno K and Kakuhata H 1998 *J. Phys. Soc. Japan* **67** 4300
[5] Konno K 1995 *Appl. Anal.* **57** 209
[6] Imai K, Konno K and Kakuhata H 1999 *J. Phys. Soc. Japan* **68** 1115
[7] Park Q-H and Shin H J 1999 *Phys. Lett.* B **454** 256
[8] Fukumoto Y and Miyazaki T 1991 *J. Fluid Mech.* **222** 369
[9] Langer J and Perline R 1991 *J. Nonlinear Sci.* **1** 71
[10] Konno K, Oono H and Kakuhata H at press